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# Gauge dependence of effective action and renormalization group functions in effective gauge theories

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## Abstract

The Caswell-Wilczek analysis on the gauge dependence of the effective action and the renormalization group functions in Yang-Mills theories is generalized to generic, possibly power counting non renormalizable gauge theories. It is shown that the physical coupling constants of the classical theory can be redefined by gauge parameter dependent contributions of higher orders in  $\hbar$  in such a way that the effective action depends trivially on the gauge parameters, while suitably defined physical beta functions do not depend on those parameters.

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# 1 Introduction

The problem of the gauge dependence of the effective action and of the renormalization group functions has been extensively studied in the mid seventies in the context of Yang-Mills theories [1, 2, 3, 4, 5]. An algebraic approach to the problem, independent of the renormalization scheme, has been proposed in [6]. On the assumption of the existence of an invariant renormalization scheme, extensions to generic, not necessarily power counting renormalizable theories have been considered in [7, 8, 9, 10] and more recently in [11, 12].

In this letter, we combine the ideas of the above cited works and reinvestigate the problem in a general setting. We clarify the essential points of the analysis by getting rid of unnecessary simplifying assumptions. More precisely:

- The analysis covers effective theories, i.e., theories that are not necessarily assumed to be power counting renormalizable. An example is Yang-Mills theory (based for simplicity on a semi-simple gauge group) involving higher dimensional gauge invariant operators as considered in [13].
- The considerations are not restricted to Yang-Mills type theories, but they extend to the case of generic reducible gauge theories with structure functions and open algebras [14, 15].
- The particular way the gauge is fixed is irrelevant. In particular, we do not need to restrict ourselves to the case of linear gauges.
- We do not assume the existence of a gauge invariant renormalization scheme.

In order to control the renormalization aspects of the problem, independently of the particular scheme being used, we assume that the quantum action principles [16, 17, 18, 19, 20] hold and follow the algebraic approach pioneered in [21, 22, 23, 24] (for reviews, see e.g. [25, 26]).

## 2 Preliminaries

## 2.1 Master equation and gauge fixing

Gauge invariance of the classical action  $S_0[\varphi^i]$  and the algebra of the gauge transformations are encoded in the minimal solution  $S[\phi^A, \phi_A^*]$  of the master equation [27, 28, 14, 15] (for reviews, see e.g. [29, 30]):

$$\frac{1}{2}(S, S)_{\phi, \phi^*} = 0. \quad (1)$$

The gauge fixing can be done in two steps: first one adds a cohomological trivial non minimal sector. This amounts to extending the minimal solution of the master equation to  $S' = S + \int d^n x B^a \bar{C}_a^*$ . The canonical BRST differential extended to the antifields and the non minimal sector is  $s = (S', \cdot)_{\phi, \phi^*}$ . The second step is to perform an anticanonical transformation generated by a gauge fixing fermion  $\Psi[\phi^A]$ : the gauge fixed action to be used for quantization is  $S_{\text{gf}}[\phi^A, \tilde{\phi}_A^*] = S'[\phi^A, \tilde{\phi}_A^* + \frac{\delta^L \Psi}{\delta \phi^A}]$ , with  $\Psi$  chosen in such a way that the propagators of the theory are well defined. For instance, in Yang-Mills type theories, standard linear gauges are obtained from

$$\Psi = \int d^n x \bar{C}_a (\partial^\mu A_\mu^a + \frac{1}{2} \xi B^a). \quad (2)$$

The cohomology of the associated BRST differential  $s = (S_{\text{gf}}, \cdot)_{\phi, \tilde{\phi}^*}$  in the space of local functions or in the space of local functionals is isomorphic to the cohomology of the canonical BRST differential in the respective spaces and can be obtained from it through the shift of antifields  $\phi^* = \tilde{\phi}^* + \delta^L \Psi / \delta \phi$ . The dependence of the gauge fixed action on the fields and antifields of the non minimal sector and their gauge fixed BRST transformations are explicitly given by

$$s \tilde{B}_a^* = \frac{\delta^R S_{\text{gf}}}{\delta B^a} = -(S_{\text{gf}}, \frac{\delta^R \Psi}{\delta B^a})_{\phi, \tilde{\phi}^*} + \tilde{C}_a^*, \quad s B_a = -\frac{\delta^R S_{\text{gf}}}{\delta \tilde{B}_a^*} = 0, \quad (3)$$

$$s \tilde{C}_a^* = \frac{\delta^R S_{\text{gf}}}{\delta \bar{C}_a} = -(S_{\text{gf}}, \frac{\delta^R \Psi}{\delta \bar{C}_a})_{\phi, \tilde{\phi}^*}, \quad s \bar{C}^a = -\frac{\delta^R S_{\text{gf}}}{\delta \tilde{C}_a^*} = -B^a. \quad (4)$$

These transformations are nilpotent and guarantee that the BRST cohomology does not depend on the fields and the antifields of the non minimal sector.

The renormalized effective action associated to the gauge fixed action  $S_{\text{gf}}[\phi, \tilde{\phi}^*]$  is denoted by  $\Gamma_{\text{gf}}[\phi_c, \tilde{\phi}^*]$ .

## 2.2 Assumptions on anomalies and couplings

Throughout the analysis, we make the following assumptions:

- The theory is stable in the sense that the BRST cohomology in ghost number 0 in the space of appropriate local functionals can be obtained by differentiation with respect to some couplings of the minimal solution of the master equation:

$$(S, A)_{\phi, \phi^*} = 0, \text{gh } A = 0 \implies A = \lambda^i \partial_{g^i} S + (S, \Xi)_{\phi, \phi^*}. \quad (5)$$

- The gauge symmetry is non anomalous in the sense that the Zinn-Justin equation

$$\frac{1}{2}(\Gamma_{\text{gf}}, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*} = 0, \quad (6)$$

holds, by adding (if necessary) finite BRST breaking counterterms to the starting point action. This is the case if the BRST cohomology in the space of local functionals in ghost number 1 is empty, or if one can prove that the corresponding anomaly candidates do not effectively arise because their coefficients vanish to all orders in  $\hbar$ .

- The couplings  $g^i$  are non redundant in the sense that

$$\mu^i \partial_{g^i} S = (S, \Xi')_{\phi, \phi^*} \implies \mu^i = 0 = (S, \Xi')_{\phi, \phi^*}. \quad (7)$$

As a consequence of this definition the value of the non-redundant couplings is fixed in terms of observables. This procedure automatically ensures that there is no mixing among physical and unphysical (or redundant) couplings (see for example [31]). However, it implies that one should rely on a specific renormalization scheme. In the following we will show that it is indeed possible to obtain gauge-parameter independent quantities without using a specific renormalization scheme.

#### Remarks:

(i) Because the gauge fixing is an anticanonical transformation, the relations (5) and (7) hold in terms of  $S_{\text{gf}}$ , local functionals  $\Xi, \Xi'$  modified through the replacement  $\phi^* \longrightarrow \tilde{\phi}^* + \delta\Psi/\delta\phi$  and the antibrackets in terms of  $\phi, \tilde{\phi}^*$ .

(ii) What the appropriate space of local functionals is precisely depends on the context. Usually it is the space of integrals of  $x^\mu$  independent polynomials or power series in the couplings, the  $dx^\mu$ , the fields, antifields and their derivatives, which can be further restricted by global symmetries such as Lorentz invariance or by power counting arguments. In particular, in the case of theories with massless and massive particle, the presence of IR singularities might restrict the space of local functionals. However, this situation can be handled by defining a proper IR power counting [26].

(iii) The last assumptions means that the coupling constants  $g^i$  are associated to independent BRST cohomological classes. It is the crucial assumption that allows to extend the Caswell-Wilczek arguments to generic gauge theories.

We assume that the  $g^i$  are the only couplings on which the minimal solution of the master equation depends. They can be considered as the “physical couplings on the classical level”. Note that in the gauge fixed theory, for any parameter  $\xi^\alpha$  appearing in  $\Psi$ , we have

$$\partial_{\xi^\alpha} S_{\text{gf}} = -(S_{\text{gf}}, \partial_{\xi^\alpha} \Psi)_{\phi, \tilde{\phi}^*}, \quad (8)$$

which means in particular that all the parameters introduced through the gauge fixing alone are redundant.

Hence, we will assume that in the gauge fixed theory, the only additional couplings besides the physical  $g^i$ 's, are the redundant gauge couplings  $\xi^\alpha$  satisfying (8). Notice that the wave function normalization constants are redundant couplings, since they can be introduced through anticanonical field antifield redefinitions.

### 3 “Physical” coupling constants on the quantum level

According to the quantum action principle,  $\partial_{\xi^\alpha} \Gamma_{\text{gf}} = K_\alpha \circ \Gamma_{\text{gf}}$ , where  $K_\alpha = -(S_{\text{gf}}, \partial_{\xi^\alpha} \Psi)_{\phi_c, \tilde{\phi}^*} + O(\hbar)$ . It follows from lemma 1 of the appendix that this implies in a first step

$$\partial_{\xi^\alpha} \Gamma_{\text{gf}} = (\Gamma_{\text{gf}}, [-\partial_{\xi^\alpha} \Psi]^Q \circ \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*} + \hbar K'_\alpha \circ \Gamma_{\text{gf}}. \quad (9)$$

Here  $[-\partial_{\xi^\alpha} \Psi]^Q$  is the renormalized operator  $\partial_{\xi^\alpha} \Psi$ . Notice that it requires a renormalization which independent from the renormalizations needed for the effective action  $\Gamma$ . In the literature [32, 33, 6], different approaches have been used to define  $[-\partial_{\xi^\alpha} \Psi]^Q$  based on the Wilson expansion or on the extended BRST technique [6]. All of these approaches amount to obtain the equation (9) where  $K'_\alpha$  can be studied algebraically.

One can then go on to show (see also the appendix) that

$$[\partial_{\xi^\alpha} + \hbar \rho_\alpha^i \partial_{g^i} + (L_\alpha \circ \Gamma_{\text{gf}}, \cdot)_{\phi_c, \tilde{\phi}^*}] \Gamma_{\text{gf}} = 0, \quad (10)$$

where  $L_\alpha = -\partial_{\xi^\alpha} \Psi + O(\hbar)$  and the coefficients  $\rho_\alpha^i$  are formal power series in  $\hbar$  depending on the couplings  $g^i$  and  $\xi^\beta$ .

Let us define  $D_\alpha = \partial_{\xi^\alpha} + \hbar \rho_\alpha^i \partial_{g^i}$ . By adapting the extended BRST technique of [6] to the present context, one can show (see appendix) that there exist local functionals  $K_{[\alpha\beta]}$  such that

$$[D_\alpha, D_\beta]^i \partial_{g^i} \Gamma_{\text{gf}} = -(K_{[\alpha\beta]} \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*}. \quad (11)$$

It then follows from the lemma 2 proved in the appendix that

$$[D_\alpha, D_\beta]^i = 0 = (K_{[\alpha\beta]} \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*}. \quad (12)$$

$[D_\alpha, D_\beta]^i = 0$  reads explicitly  $\partial_{\xi[\beta}\rho_{\alpha]}^i + \hbar\rho_{[\beta}^j\partial_{g^j}\rho_{\alpha]}^i = 0$ , which gives to lowest order in  $\hbar$  the relation  $\partial_{\xi[\beta}\rho_{0\alpha]}^i = 0$ . Using the standard Poincaré lemma (assuming that the gauge parameters space has trivial topology), there exist functions  $G_1^i(\xi, g)$  such that  $\rho_{0\alpha}^i = \partial_{\xi^\alpha}G_1^i(\xi, g)$ . Let us now define new couplings  $g_1^i = g^i - \hbar G_1^i(\xi, g)$  and the inverse transformation  $g^i = g_1^i + \hbar G_1^i(\xi, g_1) + O(\hbar^2)$ . An explicit integration formula for  $G_1^i(\xi, g)$  has been given in refs. [32, 5].

If we denote the generating functional in terms of the new couplings with a subscript 1,  $\Gamma_{1\text{gf}}(g_1, \xi) = \Gamma_{\text{gf}}(g(g_1, \xi), \xi)$  and use the same notation for all functionals, it follows that  $\partial_{\xi^\alpha}\Gamma_{1\text{gf}} + ((L_{1\alpha} \circ \Gamma_{1\text{gf}}, \Gamma_{1\text{gf}})_{\phi_c, \tilde{\phi}^*} + \hbar^2 \bar{\rho}_\alpha^i(g_1, \xi) \partial_{g_1^i} \Gamma_{1\text{gf}} = 0$ , for some  $\bar{\rho}_\alpha^i(g_1, \xi)$ . By a succession of redefinitions of the couplings  $g^i$ , we can thus achieve (dropping the subscripts)

$$\partial_{\xi^\alpha}\Gamma_{\text{gf}} + (L_\alpha \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*} = 0. \quad (13)$$

This leads to the following definition:

*The physical coupling constants  $g^i$  on the quantum level are such that the variation of the effective action with respect to the gauge parameters is given by  $(\Gamma_{\text{gf}}, \cdot)$  acting on a local insertion.*

It is the natural generalization of what one considers as physical on the classical level. It follows that physical couplings  $g^i$  on the classical level stay physical in the quantum theory, by using the additional freedom of redefinitions of the  $g^i$  by terms of higher order in  $\hbar$  involving the gauge parameters.

After projection on the physical states, equation (13) implies the gauge parameter independence of  $\Gamma_{\text{gf}}$ . Together with the BRST invariance expressed through the Zinn-Justin equation (6), these equations are the substitute for the gauge invariance of the original action. A kind of direct gauge invariance for  $\Gamma_{\text{gf}}$  can be achieved using the background field method, which will not be discussed here.

The procedure presented here differs from the conventional approach to gauge-parameter independent quantities (see for example [31]) since it does not rely on a specific renormalization scheme and on the physical observable used to fix the renormalization constants. A similar approach has been pursued in [5, 34] following the work of Zimmermann [35].

## 4 Physical beta functions in the renormalization group equation

Let us start for simplicity with the case where the theory is renormalizable by constant redefinitions of the fields and the antifields and by coupling constant redefinitions. Then, the renormalization group equation is

$$[\mu\partial_\mu + \hbar\beta^i\partial_{g^i} + \hbar\delta^\alpha\partial_{\xi^\alpha}]\Gamma_{\text{gf}} + \hbar\gamma_B^A\left(\int d^m x \tilde{\phi}_A^*\phi_c^B, \Gamma_{\text{gf}}\right)_{\phi_c, \tilde{\phi}^*} = 0. \quad (14)$$

As in [1], we inject (10) respectively (13) into (14) to get

$$[\mu\partial_\mu + \hbar\bar{\beta}^i\partial_{g^i}]\Gamma_{\text{gf}} + \hbar(C \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*} = 0, \quad (15)$$

where  $\bar{\beta}^i = \beta^i - \hbar\delta^\alpha\rho_\alpha^i$  and  $C = \int d^n x \gamma_B^A \tilde{\phi}_A^* \phi_c^B - \hbar\delta^\alpha L_\alpha$ . Note that in the second case  $\bar{\beta}^i = \beta^i$ , because  $\rho_\alpha^i = 0$ .

In the general case, it is still possible to prove directly that (15) holds, for some  $\bar{\beta}^i$  and some local insertion  $C \circ \Gamma_{\text{gf}}$  (see appendix). This leads to the following definition:

*The physical beta functions  $\bar{\beta}_i$  of the renormalization group equation are the coefficients of the derivatives  $\partial_{g^i}$  associated to physical couplings  $g^i$  of the classical level, in the renormalization group equation where the derivatives with respect to the redundant couplings have been eliminated.*

The derivation of (15) given in the appendix shows that it is always possible to cast the renormalization group equation in this form as long as the quantum action principle holds, and the theory is non anomalous and stable.

If one follows [1] and commutes the functional operators of equations (10) and (15), one gets by defining  $D = \mu\partial_\mu + \hbar\bar{\beta}^j\partial_{g^j}$ ,

$$[D, D_\alpha]^i \partial_{g^i} + (E_\alpha, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*} = 0, \quad (16)$$

where  $E_\alpha = D[L_\alpha \circ \Gamma_{\text{gf}}] - D_\alpha[\hbar C \circ \Gamma_{\text{gf}}] + (\hbar C \circ \Gamma_{\text{gf}}, L_\alpha \circ \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*}$ .

Again, we deduce  $[D, D_\alpha]^i = 0$  (see appendix). If one uses the physical couplings of the quantum level, where  $\rho_\alpha^i = 0$ , these relations reduce to

$$\partial_{\xi^\alpha} \bar{\beta}^i = 0. \quad (17)$$

This gives the main result:

*In a non anomalous stable theory, the physical  $\beta$  functions do not depend on the gauge parameters of the theory, if the effective action is expressed in terms of physical coupling constants of the quantum level.*

We also note that if one integrates the renormalization group equation

$$\mu \frac{d}{d\mu} G^i(g, \mu) = \hbar \bar{\beta}^i(G, \mu) \quad (18)$$

and replaces the couplings  $g^i$  by the running couplings  $G^i(g, \mu)$  in the effective action, equation (15) reduces to the statement that the renormalization scale dependence of the effective action is given by  $(\Gamma_{\text{gf}}, \cdot)$  acting on a local insertion,

$$\mu \frac{d}{d\mu} \Gamma_{\text{gf}} + \hbar(C \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}})_{\phi_c, \tilde{\phi}^*} = 0. \quad (19)$$

Again, after projection on the physical states, this equation expresses the renormalization scale independence of the effective action. The compatibility conditions  $[D, D_\alpha]^i = 0$  and  $[D_\alpha, D_\beta]^i = 0$  guarantee that the various redefinitions of the couplings  $g^i$  can be done simultaneously.

## 5 “Physical” effective action

Let us define the functionals

$$\Gamma'[\phi_c, \phi_c^*] = \Gamma_{\text{gf}}[\phi_c, \phi_c^* - \frac{\delta\Psi(\phi_c)}{\delta\phi_c}], \quad (20)$$

and

$$\Gamma[\phi_c, \phi_c^*] = \Gamma'[\phi_c, \phi_c^*] - \int d^n x B_c^a \bar{C}_{ac}. \quad (21)$$

In other words, we undo, after quantization, the gauge fixing on the level of the effective action. The gauge fixing procedure and the passage to  $\Gamma$  can be summarized by the following diagram:

$$\begin{aligned} S[\phi, \phi^*] &\rightarrow S'[\phi, \phi^*] = S[\phi, \phi^*] + \int d^n x B^a \bar{C}_a^* \rightarrow S_{\text{gf}}[\phi, \tilde{\phi}^*] = S'(\phi, \tilde{\phi}^* + \frac{\delta\Psi}{\delta\phi}) \\ &\quad \downarrow \\ \Gamma[\phi_c, \phi_c^*] &= \Gamma'[\phi_c, \phi_c^*] - \int d^n x B_c^a \bar{C}_{ac}^* \leftarrow \Gamma'[\phi_c, \phi_c^*] = \Gamma_{\text{gf}}[\phi_c, \phi_c^* - \frac{\delta\Psi}{\delta\phi_c}] \leftarrow \Gamma_{\text{gf}}[\phi_c, \tilde{\phi}^*] \end{aligned}$$

Because the shift in the antifields is a canonical transformation, (6) implies

$$\frac{1}{2}(\Gamma', \Gamma')_{\phi_c, \phi_c^*} = 0. \quad (22)$$

Furthermore,  $\Gamma'[\phi_c, \phi_c^*] = S'[\phi_c, \phi_c^*] + O(\hbar)$  and  $\Gamma[\phi_c, \phi_c^*] = S[\phi_c, \phi_c^*] + O(\hbar)$ . Note however that  $\Gamma'[\phi_c, \phi_c^*]$  or  $\Gamma[\phi_c, \phi_c^*]$  cannot be interpreted directly as the generating functional for 1PI vertex functions associated to  $S'[\phi, \phi^*]$ , respectively  $S[\phi, \phi^*]$ , since these actions are gauge invariant and cannot be used to derive Feynman rules. Rather, a particular Green's functions of  $\Gamma[\phi_c, \phi_c^*]$  is given by the combination of 1PI vertices of  $\Gamma_{\text{gf}}[\phi_c^A, \tilde{\phi}_A^*]$  obtained by using the chain rule of differentiation.

In the case of Yang-Mills theory with the linear gauge fixing fermion (2), the functional  $\Gamma[\phi_c, \phi_c^*]$  coincides with the reduced functional  $\hat{\Gamma}$  introduced in [2, 27]. Hence,  $\Gamma[\phi_c, \phi_c^*]$  can be considered to be the generalization of this functional to the case of generic gauge theories with possibly non linear gauge fixing.

Let us compute the dependence of  $\Gamma$  on the gauge parameters:  $\partial_{\xi^\alpha} \Gamma = \partial_{\xi^\alpha} \Gamma' - \int d^n x \frac{\delta^R \Gamma_{\text{gf}}}{\delta \tilde{\phi}_A^*} | \frac{\delta^L \partial_{\xi^\alpha} \Psi}{\delta \phi^A}$ , where  $|$  means that one has to substitute  $\tilde{\phi}^*$  by  $\phi_c^* - \frac{\partial\Psi}{\partial\phi_c}$ . Using (13) and the fact that the transformation is canonical, it follows that

$$\begin{aligned} \partial_{\xi^\alpha} \Gamma &= \partial_{\xi^\alpha} \Gamma' = (\Gamma_{\text{gf}}, F_{\text{gf}\alpha})_{\phi_c, \tilde{\phi}^*} \\ &= (\Gamma', F_\alpha)_{\phi_c, \phi_c^*}, \end{aligned} \quad (23)$$

where  $F_{\text{gf}\alpha}[\phi_c, \tilde{\phi}_c^*] = L_\alpha \circ \Gamma_{\text{gf}} + \partial_{\xi^\alpha} \Psi$  is of order at least  $\hbar$ , and  $F_\alpha[\phi_c, \phi_c^*] = F_{\text{gf}\alpha}[\phi_c, \phi_c^* - \frac{\partial\Psi}{\partial\phi_c}]$ . From its classical limit, it is also clear that  $\frac{\delta\Gamma}{\delta y_c^\Delta} = O(\hbar)$ , where  $y_c^\Delta \equiv (B_c^a, B_a^*, \bar{C}_c^a, \bar{C}_{ac}^*)$  denote the fields and antifields of the non minimal sector. Thus, the functional  $\Gamma[\phi_c, \phi_c^*]$  is independent, to order 0 in  $\hbar$ , of the gauge parameters and of the fields and antifields of the non minimal sector.



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## 6 Appendix

**Lemma 1** *The insertion of a BRST exact local functional  $(S_{\text{gf}}, \Xi)$  is equal to  $(\Gamma_{\text{gf}}, \cdot)$  applied to a local insertion, up to a local insertion of higher order in  $\hbar$ .*

$$(S_{\text{gf}}, \Xi) \circ \Gamma_{\text{gf}} = (\Gamma_{\text{gf}}, \Xi^Q \circ \Gamma_{\text{gf}}) + \hbar I \circ \Gamma_{\text{gf}}, \quad (\text{A.1})$$

where  $\Xi^Q = \Xi + O(\hbar)$  and  $I$  are local functionals.

Indeed, if  $S^\infty$  is the sum of  $S_{\text{gf}}$  and the BRST finite breaking local counterterms needed to achieve (6), the action  $S_\rho = S^\infty + \Xi\rho$  satisfies  $\frac{1}{2}(S_\rho, S_\rho) = (S_{\text{gf}}, \Xi)\rho + O(\hbar)$ , with  $\rho$  a Grassmann odd constant in ghost number 1. Applying the quantum action principle, we get  $\frac{1}{2}(\Gamma_\rho, \Gamma_\rho) = \Delta(\rho) \circ \Gamma_\rho$ . Putting  $\rho$  to zero, it follows from (6) that the local functional  $\Delta(0) = 0$ , so that  $\Delta(\rho) = \Delta'(\rho)\rho$ . Differentiation of the previous equation with respect to  $\rho$  and putting  $\rho$  to zero then implies  $(\Gamma_{\text{gf}}, \Xi^Q \circ \Gamma_{\text{gf}}) = \Delta'(0) \circ \Gamma_{\text{gf}}$ , for some local functional  $\Xi^Q = \Xi + O(\hbar)$ . At tree level, this equation implies that  $\Delta'(0) = (S_{\text{gf}}, \Xi) + O(\hbar)$ , which gives the result.

### Proof of (10) :

It follows that  $\partial_{\xi^\alpha} \Gamma_{\text{gf}} = (\Gamma_{\text{gf}}, [-\partial_{\xi^\alpha} \Psi]^Q \circ \Gamma_{\text{gf}}) + \hbar K'_\alpha \circ \Gamma_{\text{gf}}$ . Applying  $(\Gamma_{\text{gf}}, \cdot)$  using (6), we get to lowest order in  $\hbar$  the consistency condition  $(S_{\text{gf}}, K'_{\alpha 0}) = 0$ , so that (5) implies  $K'_{\alpha 0} = -\rho_{a1}^i \partial_{g^i} S_{\text{gf}} - (S_{\text{gf}}, N_{\alpha 1})$ . Using the quantum action principle under the form  $[\partial_{g^i} S_{\text{gf}}] \circ \Gamma_{\text{gf}} = \partial_{g^i} \Gamma_{\text{gf}} + \hbar I_i \circ \Gamma_{\text{gf}}$ , for a local insertion  $I_i \circ \Gamma$  and equation (A.1) again, we get  $\partial_{\xi^\alpha} \Gamma_{\text{gf}} - ([\partial_{\xi^\alpha} \Psi]^Q + \hbar N_{\alpha 1}^Q) \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}}) + \hbar \rho_{\alpha 1}^i \partial_{g^i} \Gamma_{\text{gf}} = \hbar^2 K''_\alpha \circ \Gamma_{\text{gf}}$ , and the reasoning can be pushed to higher orders.

### Proof of (11) :

We introduce Grassmann odd ghost number 1 parameters  $\lambda^\alpha$  and define  $S^e = S^\infty + \lambda^\alpha \partial_{\xi^\alpha} \Psi$ . Using (8) and  $\lambda^\alpha \lambda^\beta \frac{\partial^2 \Psi}{\partial \xi^\alpha \partial \xi^\beta} = 0$ , it follows that

$$\frac{1}{2}(S^e, S^e) - \lambda^\alpha D_\alpha S^e = \frac{1}{2}(\lambda^\alpha \partial_{\xi^\alpha} \Psi, \lambda^\beta \partial_{\xi^\beta} \Psi) + O(\hbar), \quad (\text{A.2})$$

where  $O(\hbar)$  is a local functional of order at least  $\hbar$ . Applying the quantum action principle, it follows that  $\frac{1}{2}(\Gamma^e, \Gamma^e) - \lambda^\alpha D_\alpha \Gamma^e = \frac{1}{2}(\lambda^\alpha \partial_{\xi^\alpha} \Psi, \lambda^\beta \partial_{\xi^\beta} \Psi) \circ \Gamma^e + \hbar A \circ \Gamma^e$ . Putting  $\lambda^\alpha$  to zero and using (6), it follows that  $A = \lambda^\alpha A_\alpha$ . Differentiation with respect to  $\lambda^\alpha$  and putting  $\lambda^\alpha$  to zero gives  $(\partial_{\xi^\alpha} \Psi^Q \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}}) - D_\alpha \Gamma_{\text{gf}} = \hbar A_\alpha(0) \circ \Gamma_{\text{gf}}$ . Using (10), we deduce that  $A_\alpha(0) \circ \Gamma_{\text{gf}} = (L'_\alpha \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}})$ , where  $\hbar L'_\alpha \circ \Gamma_{\text{gf}} = \partial_{\xi^\alpha} \Psi^Q \circ \Gamma_{\text{gf}} + L_\alpha \circ \Gamma_{\text{gf}}$ . If we now add to  $S^e$  the counterterm  $-\lambda^\alpha \hbar L'_{\alpha 0}$ , we can absorb the lowest order contribution  $A_\alpha(\lambda = 0)$  up to terms of second order in  $\hbar$  or of first order in  $\hbar$  and of second order in  $\lambda^\alpha$ . For the new  $\Gamma^e$ , we end up with  $\frac{1}{2}(\Gamma^e, \Gamma^e) - \lambda^\alpha D_\alpha \Gamma^e = [\frac{1}{2}\lambda^\alpha \lambda^\beta B_{[\alpha\beta]}(\lambda) + \hbar^2 \lambda^\alpha A'_\alpha(0)] \circ \Gamma^e$ , where  $B_{[\alpha\beta]}(\lambda) = (\lambda^\alpha \partial_{\xi^\alpha} \Psi, \lambda^\beta \partial_{\xi^\beta} \Psi) + O(\hbar)$ . Differentiation with respect to  $\lambda^\alpha$  and putting  $\lambda^\alpha$  to zero now gives  $(K_\alpha \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}}) = \hbar^2 A'_\alpha(0) \circ \Gamma_{\text{gf}}$ . At order 0 in  $\hbar$ , we get  $K_{\alpha 0} = \mu_\alpha^i \partial_{g^i} S_{\text{gf}} + (S_{\text{gf}}, M_\alpha)$ . But then  $K_\alpha \circ \Gamma_{\text{gf}} = \mu_\alpha^i \partial_{g^i} \Gamma_{\text{gf}} + (\Gamma_{\text{gf}}, M_\alpha^Q \circ \Gamma_{\text{gf}}) + \hbar K'_\alpha \circ \Gamma_{\text{gf}}$  and we can forget

about the first two terms, because they are annihilated by  $(\cdot, \Gamma_{\text{gf}})$ . In the same way we can get rid of the order  $\hbar$  contribution and assume that  $\hbar^2(K''_\alpha \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}}) = \hbar^2 A'_\alpha(0) \circ \Gamma_{\text{gf}}$ , which implies that the lowest order contribution to  $A'_\alpha(0)$  can be absorbed by adding suitable counterterm proportional to  $\lambda^\alpha$  and of order  $\hbar^2$ . Going on in the same way, one can achieve:

$$\frac{1}{2}(\Gamma^e, \Gamma^e) - \lambda^\alpha D_\alpha \Gamma^e = \frac{1}{2} \lambda^\alpha \lambda^\beta K_{[\alpha\beta]}(\lambda) \circ \Gamma^e, \quad (\text{A.3})$$

where  $K_{[\alpha\beta]}(\lambda) = (\lambda^\alpha \partial_{\xi^\alpha} \Psi, \lambda^\beta \partial_{\xi^\beta} \Psi) + O(\hbar)$ .

Acting with  $\lambda^\gamma D_\gamma$  on this equation, and using the same equation again, together with  $((\Gamma^e, \Gamma^e), \Gamma^e) = 0$ , we find  $(-\frac{1}{2} \lambda^\alpha \lambda^\beta K_{[\alpha\beta]}(\lambda) \circ \Gamma^e, \Gamma^e) - \frac{1}{2} \lambda^\alpha \lambda^\beta [D_\alpha, D_\beta] \Gamma^e = \frac{1}{2} \lambda^\gamma \lambda^\alpha \lambda^\beta D_\gamma [K_{[\alpha\beta]}(\lambda) \circ \Gamma^e]$ . Differentiating with respect to  $\lambda^\alpha$  and  $\lambda^\beta$  and putting  $\lambda$  to zero gives (11).

Note that (8) is equivalent to  $\lambda^\alpha \partial_{\xi^\alpha} S_{\text{gf}} = (S_{\text{gf}}, \lambda^\alpha \partial_{\xi^\alpha} \Psi)$ , which implies in particular that  $((\lambda^\alpha \partial_{\xi^\alpha} \Psi, \lambda^\beta \partial_{\xi^\beta} \Psi), S_{\text{gf}}) = 0$ , so that the right hand side of (11) starts indeed at order  $\hbar$ , as does the left hand side.

**Lemma 2** *The quantum analog of the classical condition (7) on the couplings  $g^i$  to be non redundant is*

$$\mu^i \partial_{g^i} \Gamma_{\text{gf}} = (\Gamma_{\text{gf}}, \Xi' \circ \Gamma_{\text{gf}}) \implies \mu^i = 0 = (\Gamma_{\text{gf}}, \Xi' \circ \Gamma_{\text{gf}}), \quad (\text{A.4})$$

if  $\Xi' \circ \Gamma_{\text{gf}}$  is a local insertion.

Indeed, at tree level and order 0 in  $\hbar$ , we deduce because of (7) that  $\mu_0^i = 0 = (S_{\text{gf}}, \Xi'_0)$ . It follows that  $\Xi'_0 = \lambda_0^i \partial_{g^i} S_{\text{gf}} + (S_{\text{gf}}, \Theta_0)$ , which implies  $\Xi' \circ \Gamma_{\text{gf}} = \lambda_0^i \partial_{g^i} \Gamma_{\text{gf}} + (\Gamma_{\text{gf}}, \Theta_0 \circ \Gamma_{\text{gf}}) + \hbar \Xi'' \circ \Gamma_{\text{gf}}$ . Using (6) and  $\mu_0^i = 0$ , we have  $\sum_{n \geq 1} \hbar^n \mu_n^i \partial_{g^i} \Gamma_{\text{gf}} = (\Gamma, \hbar \Xi'' \circ \Gamma_{\text{gf}})$ . We now can factorize  $\hbar$  and the reasoning can be pushed to higher orders.

**Proof of (15) :**

The quantum action principle implies  $\mu \partial_\mu \Gamma_{\text{gf}} + \hbar I \circ \Gamma_{\text{gf}} = 0$ . Applying  $(\Gamma_{\text{gf}}, \cdot)$  and using (6), we get to lowest order  $(S_{\text{gf}}, I_0) = 0$ . Using stability, this implies  $I_0 = \bar{\beta}_1^i \partial_{g^i} S_{\text{gf}} + (C_1, S_{\text{gf}})$ . By the same reasoning than above, this implies  $\mu \partial_\mu \Gamma_{\text{gf}} + \hbar \bar{\beta}_1^i \partial_{g^i} \Gamma_{\text{gf}} + \hbar (C_1 \circ \Gamma_{\text{gf}}, \Gamma_{\text{gf}}) + \hbar^2 I' \circ \Gamma_{\text{gf}} = 0$ , and the reasoning can be pushed to higher orders.

**Proof of  $[D, D_\alpha]^i = 0$  :**

Consider the parameters  $\xi^{\bar{\alpha}} = (\xi^\alpha, \mu)$ , the ghosts  $\lambda^{\bar{\alpha}} = (\lambda^\alpha, \Lambda)$  and the differentials  $D_{\bar{\alpha}} = (D_\alpha, D)$ . It then follows that (A.2) holds for the same  $S^e$  but with  $\lambda^\alpha D_\alpha$  replaced by  $\lambda^{\bar{\alpha}} D_{\bar{\alpha}}$ . The proof that  $[D_{\bar{\alpha}}, D_{\bar{\beta}}] = 0$  then proceeds exactly as before and includes the result we need, i.e.,  $[D, D_\alpha] = 0$ .

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